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Algebra 1

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Final

The grader cannot be expected to work his way through a sprawling mess of identities presented without a coherent narrative through line. If he can't make sense of it in finite time you could lose coherent narrative through line. If he can't make sense of it in finite time you could lose serious points. Coherent, readable exposition of your work is half the job in mathematics.

Problem 1 : (20 pt) Let G be a group and H a subgroup of index 2

1. **Explain what it means for a subgroup to be of index 2. Describe G/H and H/G (right and left quotient set.**
2. **Reprove then that H is a normal subgroup of G.**

Problem 2 : (20 pt)

1. **Let $< a >$ be a cyclic group of order n. Describe all the subgroup of $< a >$ and the number of each kind. You can directly use a result seen in class without reproving it.**
2. **Let the dihedral group D_n be given by elements a of order n and b of order 2, where $ba = a^{-1}b$. Show that any subgroup of $< a >$ is normal in D_n.**

Problem 3 : (20 pt) Let G be a simple group and suppose that $\phi : G \to H$ is a non-trivial group homomorphism.

1. **What does it means for G to be simple ?**
2. **Prove that ϕ trivial or injective.**

Problem 4 : (30 pt)

1. **Reprove that S_4 has no normal subgroup of order 3.**
2. **How is defined A_4 in S_4 ? What is its cardinality ?**
3. **Prove that S_4 is a semi direct product of A_4 and H, where H is also a subgroup of S_4.**

Problem 5 : (30 pt)
Let $Gl_n(\mathbb{R})$ denote the (multiplicative) group of invertible $n \times n$ matrixes with real entries. Let $SL_n(\mathbb{R})$ be the subset of matrices with determinant 1. Show that $SL_n(\mathbb{R})$ is a normal subgroup of $GL_n(\mathbb{R})$ and identify the quotient group $Gl_n(\mathbb{R})$ with a group that we know.

Problem 6 : (40 pt)
Let Q denote the quaternion group of order 8. Let $N = Z(Q)$ be its center, a normal subgroups of Q.

1. **Describe Q.**
2. **Find N and its index $[Q : N]$.**
3. **Find coset representatives for the left coset space Q/N.**
4. **What is the order of the quotient group Q/N ? How many groups up to isomorphism there is of this order ? Identify Q/N with one of those group ?**

Problem 7 : (60 pt)

Let G be a group of order p and q, where p and q are primes with $p < q$.

1. **Describe the Sylow subgroups of G.**
2. **Explain why the number of q-Sylow subgroups is coprime with q. Deduce this number and conclude about the normality of the q-Sylow subgroups.**
3. **Do the same with the the number of p-Sylows. Deduce the p-Sylow is normal when p does not divide $(q-1)$.**
4. **Prove that G is a semi-direct subgroup of q-Sylow subgroup with a p-Sylow subgroup.**
5. **If p does not divide $q-1$, show that G is the cyclic group of order pq.**
6. **Note that $U_q = \mathbb{Z}/(q-1)\mathbb{Z}$. If p divides $q-1$, show that it is a group generated by two element x and y of order q and p respectively such that $y^{-1}xy = x^{repr_{\phi(y)}}$, where $\phi : \mathbb{Z}/p\mathbb{Z} \to U_q$ is the homomorphism defining the action of H_p on H_q and $repr\phi(y)$ is some representative of the class $\phi(y)$ in $\mathbb{Z}$.**